Flapping Response of Lifting Rotor Blades to Atmospheric Turbulence

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Methods are developed for computing the flapping response of lifting rotor blades to atmospheric turbulence. The problem is of a type where a linear system with time varying parameters is subject to an input representing a nonstationary stochastic process. Since no previous quantitative solutions to this type of problem appear to be known, the methods developed in this paper are of general significance. The stochastic structure of the response is first outlined in terms of the nonstationary autocorrelation function and the double frequency power spectral density. Time averaging procedures for nonstationary stochastic processes are introduced and these concepts are then applied to a perturbation scheme for which, based on an assumed stochastic input, numerical examples are given. The first-order perturbation scheme is applicable throughout the advance ratio regime of current lifting rotors. Actual stochastic inputs to lifting rotor blades remain to be determined experimentally.

Nomenclature

t,T	= time
$x(t)$	= input sample function (stationary)
γ	= blade inertia or Lock number
$(\gamma/2)C(t)$	= blade aerodynamic damping
$(\gamma/2)K(t)$	= blade aerodynamic stiffness
$(\gamma/2)B(t)$	= multiplier for $x(t)$
$z(t)$	= input sample function (nonstationary)
$A(t)$	= multiplier for $z(t)$
c_j,k_j,b_j	= coefficients of Fourier series
$y(t)$	= response sample function
μ	= rotor advance ratio
f	= cycles per unit time
$f_0(2\pi f_0 = 1)$	= rotor revolutions per unit time
$X(f),Y(f),Z(f)$	= Fourier transforms
$H(f)$	= frequency response function
$S_x(f),S_y(f_1,f_2)$, etc.	= power spectral densities
$R_x(\tau),R_y(t_1,t_2)$, etc.	= autocorrelation functions
τ	= $t_2 - t_1$ = time difference
$\sigma_x,\sigma_y(t),\sigma_z(t)$	= standard deviations
$E[\ldots]$	= mathematical expectation of $[\ldots]$
$1/a$	= relaxation time
$\delta(\ldots)$	= Dirac delta function
$y(f,t)$	= response to input $e^{i2\pi ft}$
$y_B(f,t)$	= response to input $(\gamma/2)B(t)e^{i2\pi ft}$
$y_A(f,t)$	= response to input $A(t)e^{i2\pi ft}$
a_j,d_j,θ_j,d	= parameters in differential equation

Superscripts

*	= conjugate complex
$(\dot{\ })$	= time differentiation
$(\bar{\ })$	= time average

Subscripts

0, 1, ...	= basic response, first-order, etc. corrections

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Introduction

In the past the problem of lifting rotor dynamic loads and vibrations has been treated exclusively with deterministic analyses. Loads criteria are given in the form of certain discrete idealized events, a discrete gust with a prescribed time history of vertical or horizontal relative flow velocity, a discrete maneuver with a prescribed time history of pilot's control deflection, a discrete runway bump of a prescribed shape over which the aircraft is rolling with a prescribed velocity, etc. The first attempts to introduce probabilistic methods to lifting rotor dynamics are only very recent.[1,2] There are two general reasons for this step. First, the actual response of the structure cannot always be separated into a sequence of discrete load responses. Often the response to a preceding load has not as yet subsided at the time of a subsequent load application. Discrete load criteria are, therefore, often unrealistic and artificial. Second, statistical flight load data are best obtained by continuous loads measurements and should be compared with the results of continuous loads analyses.

In addition to these general considerations, there are a number of specific lifting rotor conditions likely to produce rather large random excursions of blade loads or motions. Many lifting rotors are capable of certain vibration modes which are only very lightly damped. Usually advancing or regressing blade lead-lag modes fall into this category. If the air turbulence in the rotor disk provides an excitation at the particular frequency of such a mode, rather large random amplitudes can occur. Another mechanism of random excitation of such a weakly damped blade lead-lag mode is through pilot's control inputs in response to atmospheric air turbulence. At high advance ratio low damping can also occur in blade flapping or flap bending, and again local turbulence in the rotor disk or atmospheric turbulence modulated by pilot's control inputs can lead to considerable random blade excursions.

Finally, three flight conditions of lifting rotor craft should be mentioned where the rotor generates high-intensity turbulence in the rotor disk leading to substantial random blade loads. The first is the transition from hovering to forward flight, and vice versa, where at advance ratios between 0.05 and 0.10 dynamic blade loads and vibrations are often severe because of strongly nonuniform inflow and turbulent wake recirculation. The second is the vertical descent in the vortex state which also causes large random blade loads.[3] The third is a condition with partial blade stall where unsteady stall phenomena in conjunction with air turbulence can produce sizeable random loads. In all of these cases it would be desirable to study the characteristics of the underlying air turbulence in the rotor disk and to explore ways of predicting the random rotor loads.

During the past years considerable experience has been accumulated in the measurement and prediction of airplane responses to continuous turbulence.[4-8] Airplanes with known dynamic response characteristics are used to measure parts of the atmospheric turbulence spectra. These spectra are then applied to predict turbulence responses of new prototype airplanes in the design stage. The bulk of the data involves the assumption of one-dimensional isotropic turbulence, where the turbulence velocities are uniform over the span and vary only in flight direction. However, recently also spatial distribution effects of continuous atmospheric turbulence on the motions of aircraft have been studied.[9,10] This refinement requires the knowledge of spatial cross-correlation functions for the turbulence velocities.

Unfortunately, the extensive work on airplane response to continuous atmospheric turbulence is not directly applicable to rotary wings. An airplane can be considered with reasonable approximation a time-invariable or constant parameter system, if one treats only small deviations from the equilibrium position. A rotary wing, however, exhibits in forward flight large periodic variations of its aerodynamic parameters. Furthermore, the time history of the wing gust loads encountered by an airplane can be considered with reasonable approximation a stationary, ergodic stochastic process, if one assumes uniform flight velocity through a region with uniform turbulence characteristics. Under the same conditions the time history of rotary wing gust loads cannot be described as a stationary stochastic process. A given fluctuation in axial inflow velocity through the rotor disk produces quite different loads depending on the azimuth position of the rotary wing.

Our literature survey has not uncovered quantitative solutions of the response of systems with time varying parameters under nonstationary stochastic loading. The few cases of nonstationary random responses treated in the literature all pertain to constant parameter systems represented by a time invariable transfer function. This is true, for example, for the response of airframes to random runway disturbances during deceleration after touchdown,[11] for the transient response to random excitation of a system originally at rest,[12] for the description of strong motion random earthquakes,[13] and for the response of spacecraft to time-varying random excitations during the launching phase.[14] The complexity of the analysis for time-varying systems is due to the fact that, except in very special cases, closed-form solutions valid over long time intervals do not exist for differential equations with variable coefficients. When it is possible to find rigorous solutions, as in the case of the Bessel differential equation, the quadrature operations required to obtain, for example, the response autocorrelation function are quite involved even for stationary random inputs.[15] Because of the scarcity of problems solved, current textbooks on applied random process theory devote only a brief section to nonstationary stochastic processes.[15-18]

Problem Formulation

In order to develop suitable methods for analyzing the response of lifting rotor blades to atmospheric turbulence without the burden of unessential complexities, the simplest problem has been selected, namely, that of rigid blade flapping assuming uniform flow through the rotor disk and neglecting unsteady aerodynamics effects. The flapping hinge is assumed to be located in the rotor center and the hub is assumed to be rigidly supported so that no cross coupling can occur (Fig. 1). The differential equation for the blade flapping angle y can then be written in the form[19,20]

$$\ddot{y} + (\gamma/2)C(t)\dot{y} + [1 + (\gamma/2)K(t)]y = (\gamma/2)B(t)x(t) = z(t) \quad (1)$$

The time unit is selected in such a way that the rotor angular velocity is one and the period of one rotor revolution is 2π. $x(t)$ is the fluctuating portion of the blade angle of attack, averaged over blade span and azimuth angle. $x(t)$ is considered to be a sample function of a stationary ergodic stochastic process and is representative of the air turbulence in the rotor disk. In a different description one could also introduce the axial inflow velocity as a stochastic variable. $(\gamma/2)B(t)$ is a time variable multiplier or modulating function for the stochastic variable $x(t)$. The time functions $C(t)$, $K(t)$, and $B(t)$ depend only on the rotor advance ratio and are periodic with period 2π. The functions $C(t)$ and $K(t)$ are shown as solid lines in Figs. 2a and 2b for an advance ratio of $\mu = 1.5$ at a tip loss factor of one. Because of the reversed flow effects on the retreating blade, these functions are nonanalytic. For the numerical examples in this paper

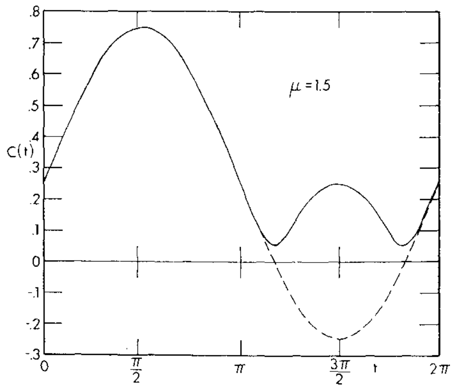


Fig. 2a Aerodynamic blade damping parameter.

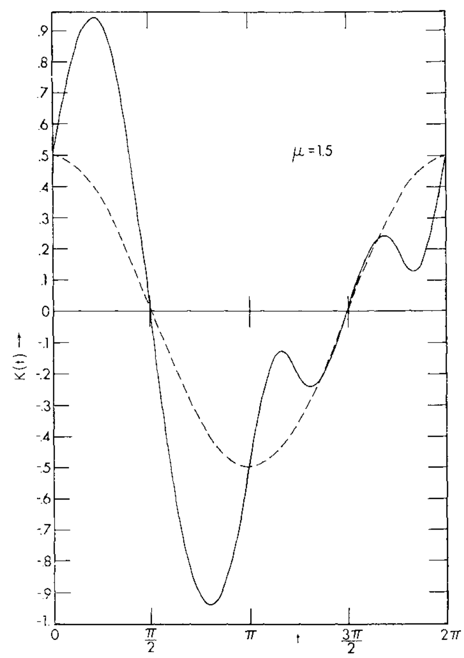
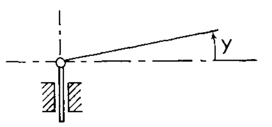


Fig. 2b Aerodynamic blade stiffness parameter.

Fig. 1 Rotor schematic.



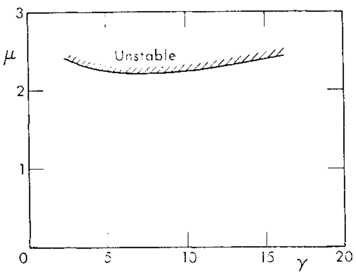


Fig. 3 Blade flapping stability limit.

simple analytical substitutes are used, namely,

$$C(t) = 0.25 + 0.33\mu \sin t \tag{2}$$

$$K(t) = 0.33\mu \cos t \tag{3}$$

which for $\mu = 1.5$, result in the dash lines of Figs. 2a and 2b. With lower advance ratio the error of the substitute functions becomes rapidly smaller and Eqs. (2) and (3) are reasonable approximations up to $\mu = 1$.

$B(t)$ also is nonanalytical and has been substituted for the numerical examples by

$$B(t) = 0.25 + 0.66\mu \sin t \tag{4}$$

which also is adequate up to $\mu = 1$. For the general methods development the existence of Fourier series for $B(t)$, $C(t)$, and $K(t)$ has been assumed.

Equation (1) has unstable solutions beyond an advance ratio of $\mu > 2.2$. The limiting advance ratio vs Lock number γ is shown in Fig. 3.[20] The substitute functions $C(t)$ result in a different stability limit.

Response Analysis for the Stationary Case

For sufficiently low advance ratio the time variable portions of $B(t)$, $C(t)$, and $K(t)$ can be neglected so that

$$\ddot{y} + (\gamma/8)\dot{y} + y = (\gamma/8)x(t) = z(t) \tag{5}$$

The input $z(t)$ now is a sample function of a stationary stochastic process. One can then easily determine the response power spectral density from the input power spectral density and vice versa.

We assume that the response Fourier transform and its inverse exist,

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft}dt \tag{6}$$

$$y(t) = \int_{-\infty}^{\infty} Y(f)e^{i2\pi ft}df \tag{7}$$

For zero initial displacement and rate of displacement it follows that $i2\pi fY(f)$ and $-(2\pi f)^2Y(f)$ are the transforms of $\dot{y}(t)$ and $\ddot{y}(t)$, respectively. Taking the Fourier transforms on both sides of Eq. (5) results in

$$Y(f) = H(f)Z(f) \tag{8}$$

where the complex frequency response function is given by

$$H(f) = 1/[-(2\pi f)^2 + (\gamma/8)i2\pi f + 1] \tag{9}$$

According to stochastic process theory the power spectral density of a weakly stationary stochastic process with sample function $z(t)$ is a real valued quantity defined by

$$S_z(f) = E[Z^*(f)Z(f)] \tag{10}$$

Applying this definition to the response $y(t)$, one obtains from Eq. (8)

$$S_y(f) = H^*(f)H(f)S_z(f) \tag{11}$$

In the time domain a random process with zero mean value is characterized by the autocorrelation function

$$R_z(t_1,t_2) = E[z(t_1)z(t_2)] \tag{12}$$

For weakly stationary random processes the autocorrelation function depends only on the time difference $\tau = t_2 - t_1$, not on t_1 or t_2 separately. The transformation from a description in the frequency domain to one in the time domain and vice versa is possible with the help of the Wiener-Khinchin relations,

$$R_z(\tau) = \int_{-\infty}^{\infty} S_z(f)e^{i2\pi f\tau}df \tag{13}$$

$$S_z(f) = \int_{-\infty}^{\infty} R_z(\tau)e^{-i2\pi f\tau}d\tau \tag{14}$$

According to Eq. (14) the power spectral density is the Fourier transform of the autocorrelation function. In Eq. (6) we have ignored certain mathematical subtleties concerning the validity of transforming a random sample function $x(t)$ in the time domain to an associated sample function $x(f)$ in the frequency domain. Nevertheless, the definition (10) for the power spectral density can formally be used as long as the autocorrelation function $R_z(\tau)$ is absolutely integrable which is the case for all practically occurring random sample functions. For details see Papoulis (Ref. 16, p. 465).

The standard deviation defined for random processes with zero mean value by

$$\sigma_z^2(t) = R_z(t,t) = E[z(t)z(t)]$$

is for a stationary stochastic process according to Eq. (13)

$$\sigma_z = [R_z(0)]^{1/2} = \left[\int_{-\infty}^{\infty} S_z(f)df\right]^{1/2} \tag{15}$$

If the random variable with zero mean value has a normal or logarithmic normal probability distribution, the power spectral density or the autocorrelation function gives a complete stochastic description of the stationary random process. Even in more general cases the first- and second-order stochastic properties as defined by $S_z(f)$ or $R_z(\tau)$ still contain the most important information about the random process.

We now assume in Eq. (5) for the average angle of attack $x(t)$ a particularly simple and frequently used random process with exponential autocorrelation function

$$R_x(\tau) = e^{-a|\tau|} \tag{16}$$

The associated two-sided power spectral density is

$$S_x(f) = 2a/(a^2 + 4\pi^2f^2) \tag{17}$$

These expressions are normalized to give with the definitions of Eq. (15) a unit standard deviation $\sigma_x = 1$. The actual power spectral density is then obtained by multiplying the normalized spectrum by σ_x^2. Figure 4 shows $R_x(\tau)$ vs τ and $S_x(f)$ vs f for $a = \frac{1}{2}$. For sufficiently large values of τ the function values $x(t)$ and $x(t + \tau)$ are uncorrelated. $1/a$ is a relaxation time for the correlation. For small values of $1/a$ the process approaches the case of white noise. In the numerical examples of this paper the value $1/a = 2$ was selected, as shown in Fig. 4. The relaxation time of 2 must

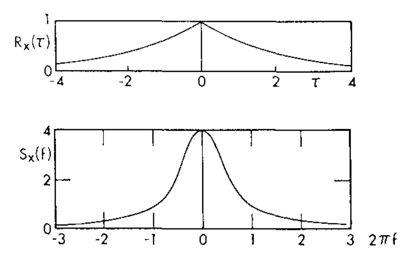


Fig. 4 Autocorrelation function and power spectral density.

be compared with the period of 2π for one rotor revolution. It is seen that a type of turbulence in the rotor disk has been assumed where the disturbing vortices are swept downstream at such a rate that they have little effect on the blades after one revolution. Although such an assumption does not seem implausible at least for some rotor operating conditions, the actual relaxation time remains to be determined experimentally.

In accordance with Eq. (11), the response power spectral density is now determined from

$$S_y(f) = H^*(f)H(f)S_x(f)\sigma_x^2(\gamma/8)^2 \quad (18)$$

where $S_x(f)$ is the normalized spectrum from Eq. (17). If the input power spectral density to Eq. (5) is derived from a stationary random process modulated by a periodic time function, the time averaged response spectrum is given by[17]

$$\bar{S}_y(f) = H^*(f)H(f)\bar{S}_z(f) \quad (19)$$

$\bar{S}_z(f)$ will be derived later [Eq. (46)]. Figure 5 shows on this basis time-averaged input and response power spectral densities for a Lock number of $\gamma = 4$ and an advance ratio of $\mu = 0.3$.

Since $\bar{S}_y(f) = \bar{S}_y(-f)$ one usually plots only the positive frequency region and then multiplies the power spectral densities by the factor two in order to retain the normalization. This was done in Fig. 5 where $2S_x(f)$ is normalized to yield

$$\sigma_x^2 = \int_0^\infty 2S_x(f)df = 1 \quad (20)$$

but where neither $\bar{S}_z(f)$ nor $\bar{S}_y(f)$ are normalized.

Response Analysis for the Nonstationary Case

We now consider the case where the time variability of $B(t)$, $C(t)$, and $K(t)$ in Eq. (1) cannot be neglected. $z(t)$ then represents a sample function of a nonstationary stochastic process. According to stochastic process theory, the power spectral density is in this case a complex valued quantity defined by[17]

$$S_z(f_1,f_2) = E[Z^*(f_1)Z(f_2)] \quad (21)$$

In order to include the definition Eq. (10) in this more general definition, one writes the power spectral density of a stationary random process in the form

$$S_z(f_1,f_2) = S_z(f_1)\delta(f_2 - f_1) \quad (22)$$

where $\delta(f)$ is the Dirac delta function with properties

$$\delta(f) = 0, \quad f \neq 0, \quad \delta(f) = \infty, \quad f = 0$$
$$\int_{-\infty}^{\infty} \delta(f)df = \int_{-\epsilon}^{\epsilon} \delta(f)df = 1, \quad \epsilon > 0 \quad (23)$$

The autocorrelation function is still defined by Eq. (12); however, now the function values depend on both t_1 and t_2, or on $\tau = t_2 - t_1$ and on $t = (t_1 + t_2)/2$. The generalized Wiener-Khinchin relations which take the place of Eq. (13) and (14) are double Fourier transforms,

$$R_z(t_1,t_2) = \iint_{-\infty}^{\infty} S_z(f_1,f_2)e^{-i2\pi(f_1t_1 - f_2t_2)}df_1df_2 \quad (24)$$

$$S_z(f_1,f_2) = \iint_{-\infty}^{\infty} R_z(t_1,t_2)e^{i2\pi(f_1t_1 - f_2t_2)}dt_1dt_2 \quad (25)$$

Following Sveshnikov[15] we introduce the response $y(f,t)$ to an excitation $e^{i2\pi ft}$, where the system is assumed to have zero displacement and rate of displacement at the origin of time. The input to Eq. (1) can be expressed by the inverse Fourier transform,

$$z(t) = \int_{-\infty}^{\infty} Z(f)e^{i2\pi ft}\,df$$

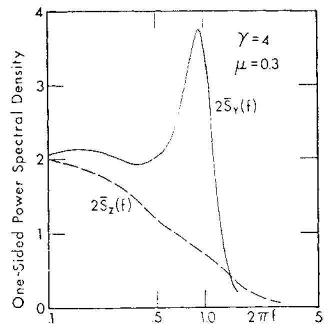


Fig. 5 Average input and response power spectral densities.

Because of the linearity of the system the response to the input $z(t)$ can be obtained from the response to $e^{i2\pi ft}$ by

$$y(t) = \int_{-\infty}^{\infty} Z(f)y(f,t)df$$

or

$$y(t) = \int_{-\infty}^{\infty} Z^*(f)y^*(f,t)df \quad (26)$$

Writing the second equation for $t = t_1$, $f = f_1$, the first for $t = t_2$, $f = f_2$, multiplying the two equations, and taking the mathematical expectations gives the response autocorrelation function

$$R_y(t_1,t_2) = \iint_{-\infty}^{\infty} \{y^*(f_1,t_1)y(f_2,t_2)S_z(f_1,f_2)df_1df_2\} \quad (27)$$

The time-dependent mean square is obtained by setting $t_1 = t_2 = t$,

$$\sigma_y^2(t) = R_y(t,t) = \iint_{-\infty}^{\infty} \{y^*(f_1,t)y(f_2,t)S_z(f_1,f_2)df_1df_2\} \quad (28)$$

These equations represent the general case of nonstationary random input to a time variable linear system.

A slight generalization of Eqs. (27) and (28) will be formulated here for the case that the input is given in the form $A(t)z(t)$, where $A(t)$ is a deterministic time function or multiplier. Since $A(t)$ can be moved across the frequency integral, we have

$$A(t)z(t) = \int_{-\infty}^{\infty} Z(f)A(t)e^{i2\pi ft}df$$

Denoting the response of the system to the input $A(t)e^{i2\pi f}$ with $y_A(f,t)$, the response is now given by

$$y(t) = \int_{-\infty}^{\infty} Z(f)y_A(f,t)df$$

and Eqs. (27) and (28) become

$$R_y(t_1,t_2) = \iint_{-\infty}^{\infty} \{y_A^*(f_1,t_1)y_A(f_2,t_2)S_z(f_1,f_2)df_1df_2\} \quad (27a)$$

$$\sigma_y^2(t) = R_y(t,t) = \iint_{-\infty}^{\infty} \{y_A^*(f_1,t_1)y_A(f_2,t_2)S_z(f_1,f_2)df_1df_2\} \quad (28a)$$

For a stationary random input to a time variable system, we have, because of Eq. (22),

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} y^*(f,t)y(f,t)S_z(f)df \quad (29)$$

For a nonstationary random input to a constant parameter system with the complex valued frequency response function $H(f)$, we have

$$y(f,t) = H(f)e^{i2\pi ft}$$

so that, by insertion in Eq. (28),

$$\sigma_y^2(t) = \iint_{-\infty}^{\infty} \{H^*(f_1)H(f_2)e^{i2\pi t(f_2 - f_1)}S_z(f_1,f_2)df_1df_2\} \quad (30)$$

This equation becomes for a stationary random input to a constant parameter system, because of Eq. (22),

$$\sigma_y^2 = \int_{-\infty}^{\infty} H^*(f)H(f)S_z(f)df \tag{31}$$

which is in agreement with Eqs. (11) and (15).

In the case of the flapping blade the input according to Eq. (1) is given by $z(t) = [\gamma/2B(t)x(t)]$, where $x(t)$ is a sample function from a stationary random process. Denoting with $y_B(f,t)$ the response to the input $(\gamma/2)B(t)e^{i2\pi ft}$, Eqs. (27a) and (28a), because of Eq. (22) reduce to the single integrals

$$R_y(t_1,t_2) = \int_{-\infty}^{\infty} y_B^*(f,t_1)y_B(f,t_2)S_x(f)df \tag{32}$$

$$\sigma_y^2 = R_y(t,t) = \int_{-\infty}^{\infty} y_B^*(f,t)y_B(f,t)S_x(f)df \tag{33}$$

Equation (33) is the same as Eq. (29) for stationary random input, except that $y(f,t)$ is replaced by $y_B(f,t)$.

For numerical work the input $e^{i2\pi ft}$ is split into $\cos 2\pi ft + i \sin 2\pi ft$, and the respective responses $y_{Bc}(f,t)$, $y_{Bs}(f,t)$ are computed separately. Equation (32) then becomes

$$R_y(t_1,t_2) = \int_{-\infty}^{\infty} \{y_{Bc}(f,t_1)y_{Bc}(f,t_2) + y_{Bs}(f,t_1)y_{Bs}(f,t_2)\}S_x(f)df \tag{34}$$

Time Averaging of Nonstationary Stochastic Processes

If a sufficiently long sample time history is given, the real-valued power spectral density of a stationary ergodic random process can easily be measured by well-known data processing means. The complex valued double frequency spectra of nonstationary random processes, however, cannot be measured directly, and are therefore of little practical use. Under certain conditions there exists a time-averaged measurable power spectral density that has some of the properties of a real-valued power spectral density of a stationary ergodic random process. As will be shown, these conditions are satisfied for the nonstationary processes represented by the input sample function $z(t)$ of Eq. (1).

We first introduce the instantaneous power spectral density defined by[21]

$$S_z(f,t) = \int_{-\infty}^{\infty} R_z\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) e^{-i2\pi f\tau}d\tau \tag{35}$$

with the associated autocorrelation function

$$R_z\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) = \int_{-\infty}^{\infty} S_z(f,t)e^{i2\pi f\tau}df \tag{36}$$

The instantaneous power spectral density $S_z(f,t)$, same as the double frequency power spectral density $S_z(f_1,f_2)$, is also in general not measurable, but it allows under certain conditions the definition of a time-averaged measurable power spectrum.

Time averaging Eq. (36), one obtains with the definitions

$$\bar{R}_z(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_z\left(t - \frac{\tau}{2}, t + \frac{\tau}{2}\right) dt \tag{37}$$

$$\bar{S}_z(f) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} S_z(f,t)dt \tag{38}$$

the relation

$$\bar{R}_z(\tau) = \int_{-\infty}^{\infty} \bar{S}_z(f)e^{i2\pi f\tau}df \tag{39}$$

In order to find out what this time averaging procedure means in terms of the double frequency spectrum $S_z(f_1,f_2)$, we substitute in Eq. (24) $t_1 = t - \tau/2$, $t_2 = t + \tau/2$ and take the time average according to Eq. (37),

$$\bar{R}_z(\tau) = \lim_{T\to\infty} \iint_{-\infty}^{\infty} \left\{S(f_1,f_2)e^{i\pi\tau(f_1+f_2)} \times \frac{\sin(f_2 - f_1)\pi T}{(f_2 - f_1)\pi T} df_1 df_2\right\} \tag{40}$$

We now assume[16] that $S_z(f_1,f_2)$ contains regular or area masses $S_r(f_1,f_2)$ and singular or line masses $S_s(f_1)\delta(f_2 - f_1)$, so that

$$S_z(f_1,f_2) = S_r(f_1,f_2) + S_s(f_1)\delta(f_2 - f_1) \tag{41}$$

Inserting this equation into Eq. (40) and considering

$$\lim_{T\to\infty} \frac{\sin(f_2 - f_1)\pi T}{(f_2 - f_1)\pi T} = \begin{matrix} 1 & \text{for } f_1 = f_2 \\ 0 & \text{for } f_1 \neq f_2 \end{matrix}$$

it is seen that the contribution of $S_r(f_1,f_2)$ to the integral in Eq. (40) is zero, while the contribution of $S_s(f_1)\delta(f_2 - f_1)$, because of Eq. (23), and setting $f_1 = f_2 = f$, is

$$\bar{R}_z(\tau) = \int_{-\infty}^{\infty} S_s(f)e^{i2\pi f\tau}df \tag{42}$$

By comparison with Eq. (39),

$$\bar{S}_z(f) = S_s(f) \tag{43}$$

This important theorem given by Papoulis[16] and henceforth referred to as Papoulis' theorem says that a real-valued time-averaged measurable power spectral density exists if the double frequency spectrum includes along the line $f_1 = f_2$ real-valued singular or line masses $S_s(f_1)\delta(f_2 - f_1)$ which are then identical with the time-averaged spectrum $\bar{S}_z(f)$.

Application to Blade Flapping Problem

We now assume for the modulating periodic time function $(\gamma\text{-}2)B(t)$ in Eq. (1) or (5) the existence of the complex Fourier series

$$\frac{\gamma}{2} B(t) = \sum_{k=-\infty}^{\infty} b_k e^{i2\pi k f_0 t} \tag{44}$$

From the definition Eq. (21) and writing the real-valued single-frequency spectrum $S_x(f)$ in the form $S_x[(f_1 + f_2)/2]\delta(f_2 - f_1)$, one obtains for $S_z(f_1,f_2)$,

$$S_z(f_1,f_2) = \sum_{k=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_k^* b_j S_x \times \left(\frac{f_1 + f_2 - f_0(k + j)}{2}\right) \delta\,[f_2 - f_1 - f_0(-k + j)] \tag{45}$$

The time-averaged spectrum, according to the Papoulis theorem, is obtained from the line masses along the line $f_1 = f_2$, so that, with $f_1 = f_2 = f$ and $k = j$,

$$\bar{S}_z(f) = \sum_{k=-\infty}^{\infty} b_k^* b_k S_x(f - kf_0) \tag{46}$$

Since $S_x(f)$ is a real-valued measurable power spectral density, the sum in Eq. (46) is also real-valued and defines a measurable time-averaged power spectral density. By inserting Eq. (46) into Eq. (39), one obtains with the substitution $f - kf_0 = g$ and with the definition

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(g)e^{i2\pi g\tau}dg$$

the associated time-averaged autocorrelation function

$$\bar{R}_z(\tau) = R_x(\tau) \sum_{k=-\infty}^{\infty} b_k^* b_k e^{i2\pi f_0 k\tau} \tag{47}$$

We now stipulate also for $C(t)$ and $K(t)$ in Eq. (1) the existence of Fourier series

$$\frac{\gamma}{2}\,C(t) = \sum_{j=-\infty}^{\infty} c_j e^{i2\pi j f_0 t} \tag{48}$$

$$\frac{\gamma}{2}\,K(t) = \sum_{j=-\infty}^{\infty} k_j e^{i2\pi j f_0 t} \tag{49}$$

The Fourier transform of Eq. (1) with f_2 as frequency domain variable is

$$-(2\pi f_2)^2 Y(f_2) + \sum_{j=-\infty}^{\infty} i2\pi f_2 c_j Y(f_2 - jf_0) + Y(f_2) + \sum_{m=-\infty}^{\infty} k_m Y(f_2 - mf_0) = \sum_{k=-\infty}^{\infty} b_k X(f_2 - kf_0) \tag{50}$$

Taking the conjugate complex value of this equation, substituting f_1 for f_2, multiplying the new equation with Eq. (50), and taking the mathematical expectation of the product leads to a lengthy functional equation between input and response power spectral densities. The right-hand side of this equation is given by Eq. (45) and describes in the $f_1 - f_2$ plane line masses along 45° lines $f_2 = f_1 \pm kf_0$, $k = 0, 1, 2, \cdots$. The left-hand side of the equation must also consist of line masses along these lines. Every term of the left-hand side has the response power spectral density in the form $S_y(f_1 - kf_0, f_2 - jf_0)$ as a factor. Measurable and of primary practical interest are only the line masses along the diagonal $f_1 = f_2$, and these line masses constitute, according to the Papoulis theorem, the time-averaged response power spectral density $\bar{S}_y(f)$. This quantity can be measured for example by the standard analog or digital techniques.[17] One could also generate a random sample function $z(t)$ representative of an input from atmospheric turbulence and then integrate the equation of motion (5) to obtain the response. In order to be meaningful, such timewise integration would have to extend over a long time period. Since usually the knowledge of the exact shape of the response function is not required, one is justified to omit the tedious determination of a sample response to a sample random input and substitute instead the determination of the average response power spectral density from the input power spectral density, which is much simpler. In engineering applications, for example, when determining the fatigue life under random loads, one usually needs only to know the power spectral density function or the autocorrelation function rather than the complete description of the random process. The concepts developed thus far will now be applied to a perturbation analysis.

Perturbation Scheme

We develop the perturbation scheme for a slight generalization of Eq. (1),

$$\overset{n}{y} + \sum_{j=1}^{n} \{a_j + d_j \sin(2\pi f_0 t + \theta_j)\} \overset{n-j}{y} = A(t)z(t) \tag{51}$$

$A(t)$ being a deterministic time function or multiplier, $z(t)$ a sample function of a nonstationary stochastic process and

$$\overset{n}{y} = \frac{d^n y}{dt^n}$$

We now introduce the operator L defining the associated constant parameter system

$$L(y) = \overset{n}{y} + \sum_{j=1}^{n} a_j \overset{n-j}{y} \tag{52}$$

If

$$|a_j| \gg |d_j| \tag{53}$$

analogous to deterministic response analyses[22,23] it is possible to introduce a perturbation parameter ϵ such that

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots \tag{54}$$

$y_0, y_1, y_2 \ldots$ all represent nonstationary random processes. Instead of solving Eq. (51), we seek a solution to the problem

$$L(y) = A(t)z(t) - \epsilon\left[\sum_{j=1}^{n} d_j \sin(2\pi f_0 t + \mu_j) y^{n-j}\right] \tag{55}$$

for $\epsilon = 1$. Substituting Eq. (54) in Eq. (55) and equating terms with the same power of ϵ, one obtains

$$L(y_0) = A(t)z(t) \tag{56}$$

$$L(y_1) = -\sum_{j=1}^{n} A_j(t) \overset{n-j}{y_0}$$
$$\vdots \tag{57}$$

$$L(y_{k+1}) = -\sum_{j=1}^{n} A_j(t) \overset{n-j}{y_k}$$
$$\vdots \tag{58}$$

where

$$A_j(t) = \sum_{j=1}^{n} d_j \sin(2\pi f_0 t + \theta_j)$$

The solutions $y_0, y_1 \ldots y_{k+1}$ are then inserted in Eq. (54) written for $\epsilon = 1$. The perturbation parameter ϵ, in our case, is merely a mathematical artifice to account for the proper order of magnitude of the various correction terms and ϵ does not appear in the solution. Now we form, from Eq. (54), the series

$$y(t_1)y(t_2) = y_0(t_1)y_0(t_2) + \epsilon\{y_0(t_1)y_1(t_2) + y_0(t_2)y_1(t_1)\} + \epsilon^2 \ldots$$

The expectation of the preceding series gives

$$R_y(t_1,t_2) = R_{y_0}(t_1,t_2) + \epsilon\{R_{y_0y_1}(t_1,t_2) + R_{y_1y_0}(t_1,t_2)\} + \epsilon^2 \ldots \tag{59}$$

A direct double Fourier transform on Eq. (59) yields

$$S_y(f_1,f_2) = S_{y_0}(f_1,f_2) + \epsilon[S_{y_0y_1}(f_1,f_2) + S_{y_1y_0}(f_1,f_2)] + \epsilon^2 \ldots \tag{60}$$

In order to compute the response autocorrelation function from Eq. (27a), one can use for the deterministic response $y_A(f,t)$ to the input $A(t)e^{i2\pi ft}$ the same perturbation scheme,

$$L(y_{A_0}) = A(t)e^{i2\pi ft} \tag{61}$$

$$L(y_{A_1}) = -\sum_{j=1}^{n-1} A_j(t) \overset{n-j}{y_{A_0}}, \text{ etc.} \tag{62}$$

and

$$y_A(f,t) = y_{A_0}(f,t) + \epsilon y_{A_1}(f,t) + \ldots \tag{63}$$

Since the linear operator L contains only constant coefficients, the solutions of Eq. (61), (62), etc. can often be given in closed form. Inserting Eq. (63) in Eq. (27a) and comparing the factors of the various powers of ϵ with Eq. (59), one finds

$$R_y(t_1,t_2) = \iint_{-\infty}^{\infty} y_{A_0}{}^*(f_1,t_1) y_{A_0}(f_2,t_2) S_z(f_1,f_2) df_1 df_2 \tag{64}$$

$$R_{y_0y_1}(t_1,t_2) = \iint_{-} y_{A_0}{}^*(f_1,t_1) y_{A_1}(f_2,t_2) S_z(f_1,f_2) df_1 df_2$$
$$\vdots \tag{65}$$

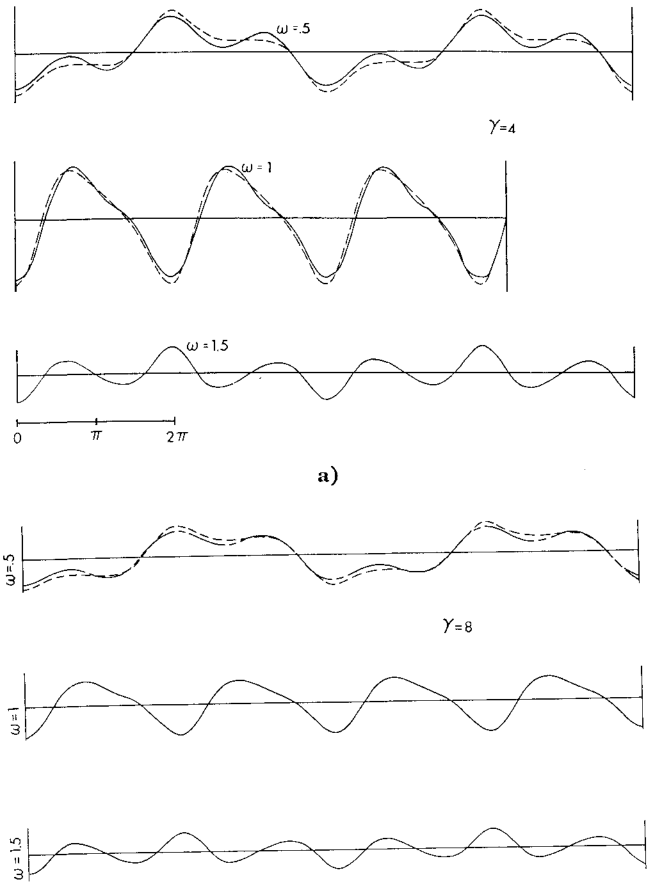


Fig. 6 Steady-state response comparisons, d = 1.

Inserting these expressions in Eq. (59) and setting $\epsilon = 1$, one gets the response autocorrelation function $R_y(t_1,t_2)$. In the application to the blade flapping problem the double integrals reduce to single integrals; see Eqs. (22) and (23). In the numerical examples only the first-order correction terms of the perturbation scheme were included.

The response power spectral density could be obtained from the Fourier transform of the autocorrelation function. However, a direct computation from Eq. (60) using the Fourier transforms of Eq. (56), (57), etc. is possible. In Eq. (60), $S_{y_0y_1}(f_1,f_2)$ is the complex valued cross-spectral density defined by

$$S_{y_0y_1}(f_1,f_2) = E[Y_0{}^*(f_1)Y_1(f_2)] \tag{66}$$

The physically realizable time-averaged spectral density $\bar{S}_y(f)$ is according to the Papoulis theorem obtained from Eq. (60) by letting $f_1 = f_2 = f$;

$$\bar{S}_y(f) = \bar{S}_{y0}(f) + 2\epsilon \,\text{real}[S_{y_0y_1}(f,f)] + \epsilon^2 \ldots \tag{67}$$

If $H(f)$ is the frequency response function corresponding to the constant parameter system defined by Eq. (52), and if we denote the input power spectral density corresponding to $A(t)z(t)$ by $S_{Az}(f_1,f_2)$, the zero-order response power spectral density from Eq. (56) is

$$\bar{S}_{y0}(f_1,f_2) = H^*(f_1)H(f_2)S_{Az}(f_1,f_2) \tag{68}$$

The time-averaged zero-order response power spectral density then is obtained by letting $f_1 = f_2 = f$;

$$\bar{S}_{y0}(f) = H^*(f)H(f)\bar{S}_{Az}(f) \tag{69}$$

For the cross-spectral density $S_{y_0y_1}(f_1,f_2)$ one finds, from Eq. (66) and from the Fourier transform of Eq. (57),

$$S_{y_0y_1}(f_1,f_2) = H(f_2)\Bigg[\frac{1}{2}\,S_{y0}(f_1,f_2 - f_0) \times \sum_{j=1}^{n} e^{i\theta j}d_j(i)^{n-j-1}(f_2 - f_0)^{n-j} - \frac{1}{2}\,S_{y0}(f_1,f_2 + f_0) \times \sum_{j=1}^{n} e^{-i\theta j}d_j(i)^{n-j-1}(f_2 + f_0)^{n-j}\Bigg] \tag{70}$$

The time-averaged response power spectral density is obtained from Eq. (67) by adding to $\bar{S}_{y0}(f)$ from Eq. (69) twice the real part of Eq. (70). Higher-order terms could be considered. The examples presented here have been computed with only the first-order correction term.

Numerical Examples

For the numerical examples Eqs. (1–4) were used. The right-hand side corresponds to values of $\gamma = 4$ and $\mu = 0.3$. In the left-hand side the value of $\gamma = 4$ was combined with variations in μ in order to obtain the effects of varying the time-dependent terms in the damping and stiffness coefficients. Thus, the equation which was numerically evaluated reads

$$\ddot{y} + (\gamma/8 + d\,\sin t)\dot{y} + (1 + d\,\cos t)y = (0.5 + 0.4\,\sin t)x(t) \tag{71}$$

where $d = \gamma\mu/6$.

As mentioned before, $X(t)$ was assumed to be a sample function of a stationary stochastic process with exponential autocorrelation function and a relaxation time of 2, which is 0.32 times the period of rotor revolution. The one-sided power spectral density is, then, according to Eq. (17),

$$2S_x(f) = 2/(0.25 + 4\pi^2 f^2) \tag{72}$$

For the numerical examples higher than first-order correc-

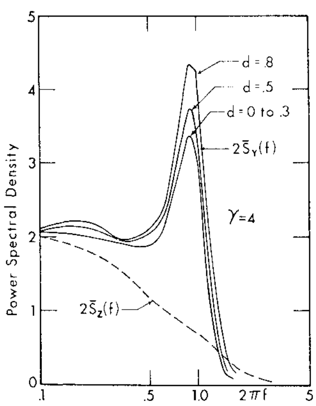


Fig. 7 Effect of d on average response power spectral density.

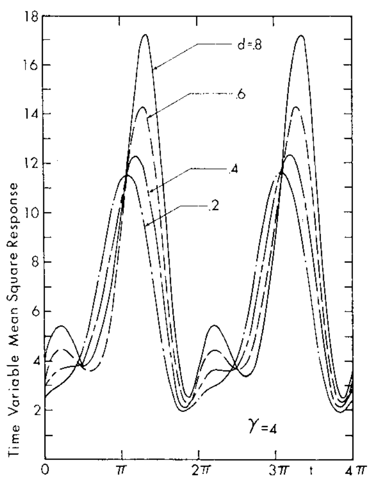


Fig. 8 Effect of d on mean-square response.

tions were in most cases found to be insignificant. To check the accuracy of the perturbation method the steady-state response to the input $\cos 2\pi ft$ was compared to the response obtained by numerical integration with the Runge-Kutta method. Figures 6a and 6b show that the error of the perturbation method with $d = 1$ and for $\gamma = 4$ and $\gamma = 8$ is quite small for all three frequencies checked, $2\pi f = 0.5$, 1.0, and 1.5. From Figs. 6a and 6b, it appears that the first-order correction with the perturbation method should yield satisfactory accuracy up to $\mu = 1$, which is the upper limit of approximate validity of Eq. (1) anyway.

Figure 7 shows for $\gamma = 4$ the effect of the parameter d on the response power spectral density. The time-averaged input power spectral density $\bar{S}_z(f)$ shown in dash lines is computed from Eq. (46), the time-averaged response power spectral density $\bar{S}_y(f)$ is computed from Eq. (67). The very remarkable result is that up to $d = 0.3$ ($\mu = 0.45$) the correction terms are for practical purposes negligible. This means that within the usual range of helicopter advance ratios nonstationary characteristics need only be considered in the time-averaged input spectrum [Eq. (46)], whereas the time-averaged response spectrum can be obtained very simply from Eq. (69) with the help of the time invariable frequency response function $H(f)$ associated with the constant parameter system defined by Eq. (52).

Figure 8 shows the effects of the parameter d and of the nonstationarity in the input on the time variable mean-square response computed with Eqs. (59, 64, and 65). Large fluctuations in the mean-square response even for quite small values of d are essentially due to the nonstationarity in the input. The ordinates in Fig. 8, to be in conformity with the input shown in Fig. 7, should be scaled by $1/4\pi$.

Concluding Remarks

For the blade flapping problem the first-order corrections from the perturbation scheme seem adequate to compute the time-averaged response power spectral density and the time variable response mean square up to surprisingly large time variations of the system parameters, certainly covering current helicopter flight regimes. With the help of the perturbation analysis it should also be possible to use blade flapping response measurements in order to gain some insight into the character of the underlying turbulence in the rotor disk, in other words, to compute the unknown input power spectral density. For advance ratios in excess of one the perturbation scheme is not adequate. An extension of the present blade flapping problem to higher advance ratios can be carried out in two phases: first, one needs to represent more accurately than was done in Eqs. (2–4) the time variable damping, stiffness, and the input modulating functions and to Fourier analyze these functions; second, one needs to compute the mean-square response by using Eq. (32). The necessary deterministic responses have now to be computed by some numerical procedure, like the Runge-Kutta method.

The random flapping response of an isolated blade to one-dimensional isotropic turbulence in the rotor disk is, of course, only the simplest problem of random phenomena in lifting rotors. The preceding study must be extended to include several blades and their interactions with each other, with the elastic pylon, and with rigid body motions. Blade flexibility must be included and the assumption of one-dimensional isotropic turbulence must be relaxed to include spatial distribution effects of the air turbulence in the rotor disk. Finally, the stochastic analysis presented herein must be extended to problems of threshold crossings and peak distributions.

References

[1] Dalzek, J. F. et al., "Studies of the Structural Response of Simulated Helicopter Rotor Blades to Random Excitation," Interim Reports I and II, AROD Contract 375, 1965 and 1966.

[2] Rich, M. J., Israel, M. H., and Kenigsberg, I. J., "Power Spectral Density Analysis of V/STOL Aircraft Structures," Paper 222, 1968, American Helicoptor Society.

[3] Yaggy, P. F. and Mort, K. W., "Wind-Tunnel Tests of Two VTOL Propellers in Descent," TN D-1766, 1963, NASA.

[4] Houbolt, J. C., Steiner, R., and Pratt, K. G., "Dynamic Response of Airplanes to Atmospheric Turbulence," TR R-199, 1964, NASA.

[5] Dempster, J. B. and Bell, C. A., "Summary of Flight Load Environmental Data Taken on B-52 Fleet Aircraft," *Journal of Aircraft*, Vol. 2, No. 5, Sept.-Oct. 1965, pp. 398–406.

[6] Steiner, R. and Pratt, K. G., "Some Applications of Power Spectra to Airplane Turbulence Problems," *Journal of Aircraft*, Vol. 4, No. 4, July-Aug. 1967, pp. 360–365.

[7] Dempster, J. B. and Roger, K. L., "Evaluation of B-52 Structural Response to Random Turbulence with Stability Augmentation Systems," *Journal of Aircraft*, Vol. 4, No. 6, Nov.-Dec. 1967, pp. 507–512.

[8] Firebaugh, J. M., "Evaluation of a Spectral Gust Model Using VGH and V-G Flight Data," *Journal of Aircraft*, Vol. 4, No. 6, Nov.-Dec. 1967, pp. 518–525.

[9] Fuller, J. R., "A Procedure for Evaluating the Spacewise Variations of Continuous Turbulence on Airplane Responses," *Journal of Aircraft*, Vol. 5, No. 1, Jan.-Feb. 1968, pp. 49–52.

[10] Swain, R. L. and Connors, A. J., "Gust Velocity Spatial Distribution Effects on Lateral-Directional Response of VTOL Aircraft," *Journal of Aircraft*, Vol. 5, No. 1, Jan.-Feb. 1968, pp. 53–59.

[11] Fung, Y. C., "The Analysis of Dynamic Stresses in Aircraft Structures During Landing as Non-Stationary Random Process," *Journal of Applied Mechanics*, Vol. 22, 1955, pp. 449–457.

[12] Caughey, T. K. and Stumpf, H. J., "Transient Response of a Dynamic System Under Random Excitation," *Journal of Applied Mechanics*, Vol. 28, Series E, No. 4, Dec. 1961, pp. 563–566.

[13] Amin, M. and Ang, A. H.-S., "Nonstationary Stochastic Model of Earthquake Motions," *Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers*, Vol. 94, No. EM2, April, 1968, pp. 559–583; also Paper 5906, American Society of Civil Engineers.

[14] Piersol, A. G., "Spectral Analysis of Non-Stationary Spacecraft Vibration Data," CR-341, 1965, NASA.

[15] Sveshnikov, A. A., *Applied Methods of the Theory of Random Functions*, 1st ed., Pergamon Press, New York, 1966, Chap. II, pp. 134–139.

[16] Papoulis, A., *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York, 1965, Chap. XII, pp. 430–450.

[17] Bendat, J. S. and Piersol, A. G., *Measurement and Analysis of Random Data*, Wiley, New York, 1966, Chap. IX, pp. 333–372.

[18] Lin, Y. K., *Probabilistic Theory of Structure Dynamics*, McGraw-Hill, New York, 1967, Chap. V, pp. 129–142.

[19] Hohenemser, K. H., "Some Aerodynamic and Dynamic Problems of the Compound Rotary-Fixed Wing Aircraft," *Proceedings of the 8th Annual Forum of the American Helicopter Society*, 1952, pp. 147–174.

[20] Sissingh, G. J., "Dynamics of Rotors Operating at High Advance Ratios," *Journal of the American Helicopter Society*, Vol. 13, No. 3, 1968, pp. 56–63.

[21] Page, G. G., "Instantaneous Power Spectra," *Journal of Applied Physics*, Vol. 23, 1952, pp. 103–106.

[22] Bellman, R., *Perturbation Methods in Mathematics, Physics and Engineering*, Holt, Rinehart, and Winston, New York, 1966, pp. 2–18.

[23] Kizner, W., "A High Order Perturbation Theory Using Rectangular Coordinates," *Celestial Mechanics and Astrodynamics*, edited by V. G. Szebehely, Academic Press, New York, 1964, pp. 81–98.